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USE OF SIMILARITY HYPOTHESES FOR SOLVING THE PROBLEM OF HOMOGENEOUS TURBULENCE DECAY

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The feasibility of defining a homogeneous, and in particular, an isotropic turbulence, using only the hypotheses of similarity of spectral, correlation, or structural functions is considered. Hypotheses of similarity ("self-preservation") of correlation functions were used in [1] and in many subsequent works in conjunction with other assumptions. Sedov [2], using only these hypotheses for analyzing the problem of decay of homogeneous and isotropic turbulent motions of a viscous incompressible fluid, had shown possible alternatives: either the energy of turbulent velocity pulsations decrease in inverse proportion to time and the linear scale increases in proportion to the square root of time, when the determination of dimensionless correlation functions requires additional assumptions; or the complete solution can be derived (and was derived by Sedov) without having to resort to supplementary assumptions. It is shown here that this solution is in good agreement with experimental data on the decay of turbulent motions of air and water downstream of a grid. With the use of the hypothesis of self-preservation of spectral tensors Sedov's solution is extended to homogeneous nonisotropic turbulence, and with the use of hypotheses of self-preservation of structural functions isotropic turbulence. In both cases alternatives similar to those of Sedov were obtained.

1. Let us consider the averaged homogeneous turbulent motion of a viscous incompressible fluid of constant density on the assumption that the pulsating motion is defined by the system of Navier-Stokes equations. The principal object of this investigation is the two-point tensor of velocity correlation

$$b_{i,j}(r_n, t) = \langle u_i(x_n, t)) u_j(x_n + r_n, t)\rangle$$

where $u_i(x_n, t)$ are velocity components at the point defined by Cartesian coordinates x_n; i, j, m and n are subscripts which assume the values 1, 2, 3; angle brackets denote an averaging operation, and $\langle u_i(x_n, t)\rangle = 0$.

For the tensor $b_{i,j}$ from the system of Navier-Stokes equations we can derive (see, e.g. [3]) the following equation:

$$\frac{\partial b_{i,j}}{\partial t} - 2\nu \frac{\partial^2 b_{i,j}}{\partial r_m \partial r_m} = S_{i,j} \tag{1.1}$$

The tensor $S_{i,j}(r_n, t)$ in Eq. (1.1) is defined in terms of third moments of velocity and moments containing pressure. Tensor $b_{i,j}(r_n, t)$ must in addition satisfy the following system of equations (see, e.g., [3]):

$$\frac{\partial}{\partial r_i} b_{i,j} = 0$$
$$b_{i,j}(r_n, t) = b_{j,i}(-r_n, t) \tag{1.2}$$

and the obvious inequality

$$\left\langle \left| \sum_{k=1}^{3} \sum_{p=1}^{l} \alpha_p z_k u_k (x_n^{(p)}) \right|^2 \right\rangle = \sum_{k,j=1}^{3} \sum_{p,s=1}^{l} \alpha_p \bar{\alpha}_s z_k \bar{z}_j b_{k,j} (r_n^{(p,s)}) \geqslant 0 \tag{1.3}$$
$$(p, s = 1, 2, \ldots, l; \quad n, k = 1, 2, 3)$$

in which α_p and z_k are arbitrary complex numbers; the vinculum denotes the operation of complex conjugation; $r_n^{(p,s)}$ is the difference between the coordinates $x_n^{(s)} - x_n^{(p)}$ of arbitrary l points of space, and l is an arbitrary number of positive integers.

We introduce the tensor of the three-dimensional energy spectrum and the spectral tensor related to tensor $S_{k,j}(r_n, t)$ by

$$b_{k,j}(r_n, t) = \iiint_{-\infty}^{\infty} E_{k,j}(k_n, t)\, e^{i(k_m r_m)}\, dk_1 dk_2 dk_3 \tag{1.4}$$

$$S_{k,j}(r_n, t) = \iiint_{-\infty}^{\infty} F_{k,j}(k_n, t)\, e^{i(k_m \cdot r_m)}\, dk_1 dk_2 dk_3 \tag{1.5}$$

From Eq. (1.1) we now obtain

$$\frac{\partial}{\partial t} E_{i,j} + 2\nu k^2 E_{i,j} = F_{i,j} \quad (k^2 = k_m k_m) \tag{1.6}$$

and from the system of relationships (1.2) and (1.3)

$$E_{i,j} k_i = 0 \tag{1.7}$$
$$E_{i,j}(-k_n) = \overline{E_{i,j}}(k_n) = E_{j,i}(k_n)$$

and the inequality

$$E_{ij}(k_n, t)\; z_i \bar{z}_j \geqslant 0 \tag{1.8}$$

which is valid for any complex numbers z_1, z_2 and z_3.

The last inequality follows from Bochner's theorem [4] about the representation of a positive definite function (function $z_k \bar{z}_j b_{k,j}(r_n)$) is such a function in virtue of relationships (1.2) and (1.3)) in the form of a Fourier-Stiltjes integral.

It follows from the same theorem that tensor $b_{i,j}(z_n, t)$ can be always presented in the form of a Fourier-Stiltjes integral. Interoduction of the spectrum by equality (1.4) is based on the assumption that this Fourier-Stiltjes integral can be represented in the form of a Fourier integral. To introduce tensor $F_{i,j}(k_n, t)$ by equality (1.5) and to derive Eq. (1.6) and the system of equations (1.7) it is sufficient to assume the absolute convergence of integrals

$$\iiint_{-\infty}^{\infty} E_{i,j} k_m dk_1 dk_2 dk_3, \quad \iiint_{-\infty}^{\infty} E_{i,j} k_m k_n dk_1 dk_2 dk_3$$
$$\iiint_{-\infty}^{\infty} \frac{\partial}{\partial t} E_{i,j} dk_1 dk_2 dk_3$$

The general form of tensor $E_{i,j}(k_n, t)$, which satisfies the system of relationships (1.7) and (1.8) [5] is

$$E_{i,j} = E_i \bar{E}_j + E' (\delta_{ij} - k_i k_j / k^2) \tag{1.9}$$

where $E'(k_n, \ t)$ is an arbitrary nonnegative function and $E_i(k_n, \ t)$ is an arbitrary complex vector function which satisfies the equation $E_j k_j = 0$.

The problem is thus reduced to solving Eq. (1.6) for tensor $E_{i,j}(k_n, \ t)$ which may be specified in the form (1.9). Since Eq. (1.6) contains tensor $F_{i,j}(k_n, \ t)$, hence this equation is open, and its closure requires additional hypotheses.

2. Let us assume that there exist functions $l(t)$ and $b(t)$ whose dimensions are, respectively, length and the square of velocity, and dimensionless tensors ψ_{ij} and f_{ij}, which depend on dimensionless arguments $\eta_n = k_n l$, such that for $\eta' < k_n l < \eta''$ and $t > t'$ the equalities

$$E_{i,j}(k_n, \ t) = bl^3 \psi_{ij}(\eta_n) \qquad F_{i,j}(k_n, \ t) = b^{3/2} l^2 f_{ij}(\eta_n) \tag{2.1}$$

are satisfied.

If the hypotheses of spectral tensor similarity (2.1) are satisfied for all k_n ($\eta' = -\infty, \eta'' = +\infty$), then similar equalities are valid for correlation tensors. In the isotropic case these equalities are equivalent to the Kármán-Howarth hypotheses on the self-conservation of correlation functions. Substituting expressions (2.1) into Eq. (1.6) and passing from variables k_n, t to η_n, t, we obtain the unique differential equation

$$\left(3\psi_{ij} + \eta_m \frac{\partial \psi_{ij}}{\partial \eta_m}\right) bl^2 \frac{dl}{dt} + \psi_{ij} l^3 \frac{db}{dt} + 2\nu bl\eta^2 \psi_{ij} = l^2 b^{3/2} f_{ij} \tag{2.2}$$

which links tensors $\psi_{ij}(\eta_n)$, $f_{ij}(\eta_n)$, and functions $b(t)$ and $l(t)$.

Let us analyze this equation.

Dividing Eq. (2.2) by $l^2 b^{3/2}$ and differentiating with respect to t, we obtain

$$\left(3\psi_{ij} + \eta_m \frac{\partial \psi_{ij}}{\partial \eta_m}\right) \frac{d}{dt}\left[b^{-1/2} \frac{dl}{dt}\right] + \psi_{ij} \frac{d}{dt}\left[l b^{-3/2} \frac{db}{dt}\right] + \eta^2 \psi_{ij} \frac{d}{dt}\left[\frac{2\nu}{l\sqrt{b}}\right] = 0 \tag{2.3}$$

Functions $\psi_{ij}\eta^2$ and ψ_{ij} are obviously linearly independent (subscripts i and j are fixed). Hence two possible cases must be considered in relation with Eq. (2.3):

1) Tensors

$$\psi_{ij}, \ \eta_m \partial \psi_{ij} / \partial \eta_m, \ \eta^2 \psi_{ij} \tag{2.4}$$

admit one and only one linear relationship with constant coefficients

$$a_1 \left(3\psi_{ij} + \eta_m \frac{\partial \psi_{ij}}{\partial \eta_m}\right) - a_2 \psi_{ij} + a_3 \psi_{ij} \eta^2 = 0 \tag{2.5}$$

where $a_1^2 + a_2^2 + a_3^2 \neq 0$.

2) Relationship (2.5) between tensors (2.4) can only be satisfied if $a_1 = a_2 = a_3 = 0$.

In this case from Eq. (2.3) follows that

$$\frac{d}{dt}\left[b^{-1/2} \frac{dl}{dt}\right] = \frac{d}{dt}\left[l b^{-3/2} \frac{db}{dt}\right] = \frac{d}{dt}\left[\frac{2\nu}{l\sqrt{b}}\right] = 0$$

from which we obtain the laws of decay of $b(t)$ and $l(t)$

$$l = c' (t + t_0)^{1/2}, \quad b = c'' (t + t_0)^{-1} \tag{2.6}$$

For ψ_{ij} and f_{ij} we now have from (2.2) a single equation with constant coefficients. The determination of these tensors necessitates supplementary assumptions.

Let us revert to case (1).

We pass from Cartesian coordinates η_1, η_2, η_3 to spherical coordinates η, θ, φ defined by

$$\eta_1 = \eta \cos\theta, \quad \eta_2 = \eta \sin\theta \cos\varphi, \quad \eta_3 = \eta \sin\theta \sin\varphi$$

We now have

$$\eta_m \frac{\partial}{\partial \eta_m} \psi_{ij}(\eta_1, \eta_2, \eta_3) = \eta \frac{\partial}{\partial \eta} \psi_{ij}(\eta, \theta, \varphi)$$

For $a_3 = 0$ from (2.5) we have

$$\psi_{ij} = c_{ij}(\theta, \varphi)\, \eta^{-3+a_2/a_1}$$

and for $a_2 + 2a_1 \neq 0$ from Eq. (2.3) we obtain for $b(t)$ and $l(t)$ the same decay laws (2.6) as in case (2).

Henceforth we assume that $a_3 \neq 0$.

Let us set $a_3 = 2$. From (2.5) and (2.3) we have

$$\frac{1}{2}\frac{d}{dt}\left[\frac{2\nu}{l\sqrt{b}}\right] = \frac{1}{a_1}\frac{d}{dt}\left[b^{-1/2}\frac{dl}{dt}\right] = -\frac{1}{a_2}\frac{d}{dt}\left[lb^{-3/2}\frac{db}{dt}\right]$$

from which we obtain

$$b^{-1/2}\frac{dl}{dt} = a_1\frac{\nu}{l\sqrt{b}} + p, \quad lb^{-3/2}\frac{db}{dt} = -a_2\frac{\nu}{l\sqrt{b}} + q \tag{2.7}$$

where p and q are constants of integration. Substituting (2.7) into (2.2) and allowing for (2.4), we obtain

$$f_{ij} = (3p + q)\psi_{ij} + p\eta\, \partial\psi_{ij} / \partial\eta \tag{2.8}$$

Eliminating time from the system of equations (2.7) and integrating the derived equation, for $a_2 \neq 2a_1$, $q \neq -2p$, and $a_2 p + a_1 q \neq 0$ we obtain

$$\nu l^{-1} b^{-1/2} = C l^{q/\gamma} b^{-p/\gamma} + \frac{2p+q}{a_2 - 2a_1} \tag{2.9}$$
$$\gamma = 2(pa_2 + qa_1) / (a_2 - 2a_1)$$

where C is a constant of integration. Using (2.9) we can integrate the first equation of system (2.7) by the method of quadrature

$$t + t^* \frac{2}{\nu(a_2 - 2a_1)} \int_{z_0}^{z} z^{-1+2\gamma/(2p+q)} \left[C\frac{z}{\nu} + \frac{2p+q}{\nu(a_2 - 2a_1)}\right]^{-4p/(2p+q)} dz \tag{2.10}$$
$$(z = l^{q/\gamma} b^{-p/\gamma})$$

Formulas (2.9) and (2.10) provide the solution for the functions $b(t)$ and $l(t)$.

For $a_3 = 2$ from (2.4) and (2.8) we have

$$\psi_{ij} = c_{ij}(\theta, \varphi)\, e^{-\eta^2/a_1} \eta^{-3+a_2/a_1} \tag{2.11}$$

$$f_{ij} = \left(p\frac{a_2}{a_1} + q - \frac{2p}{a_1}\eta^2\right)\psi_{ij} \tag{2.12}$$

Coefficients $c_{ij}(\theta, \varphi) = c_{i,j}(\eta_n / \eta)$, serve as initial conditions and must satisfy a system of algebraic equations similar to system (1.7), (1.8) and derived from the latter. Using the general form of the spectral tensor (1.9), we solve the system of equations for c_{ij}

$$\psi_{ij}(\eta_n) = e^{-\eta^2/a_1}\eta^{-3+a_2/a_1}\left[\psi_i\overline{\psi}_j + \psi\left(\delta_{ij} - \frac{\eta_i\eta_j}{\eta^2}\right)\right] \tag{2.13}$$

where $\psi(\eta_n / \eta)$ is an arbitrary nonnegative function and $\psi_i(\eta_n / \eta)$ is an arbitrary complex vector function which satisfies equation $\psi_j\eta_j = 0$. Arguments η_1 / η, η_2 / η and η_3 / η of these functions are linked by the relationship $(\eta_m / \eta)(\eta_m / \eta) = 1$.

Let us restate our conclusions taking the laws of decay of $b(t)$ and $l(t)$ as the basis. Thus, if the hypotheses (2.1) about the self-preservation of spectral tensors are satisfied by the decay of homogeneous turbulent motions of a viscous incompressible fluid, then either laws (2.6) are satisfied for b and l and the determination of the dimensionless spectral tensors ψ_{ij} and f_{ij} requires additional assumptions, or, if laws (2.6) are not satisfied, $b(t)$ and $l(t)$ are determined by formulas (2.9) and (2.10) and the spectral tensors ψ_{ij} and f_{ij} by formulas (2.13) and (2.12).

If for reasonably great t the hypotheses (2.1) are satisfied throughout space k_n, we can pass from spectral to correlation tensors. From Eq. (2.8) we can derive

$$lb^{-1/2}S_{ij} = -pr_n\frac{\partial}{\partial r_n}b_{i,j} + qb_{i,j}$$

From this with allowance for the equality $S_{j,j}(0, t) = 0$ (see [3]) follows that $q = 0$. For $q = 0$ system (2.7) coincides with the system of equations for $b(t)$ and $l(t)$ which was obtained and solved for isotropic turbulence [2]. As shown by Sedov, in this case $a_1 > 0$ and $a_2 > 0$, and $a_1 = 0.5$ can be assumed. Following Sedov we introduce the notation $a_2 = 10\alpha$. For $\alpha \neq 0.1$ the solution of system (2.7) can be written as

$$l = \nu b^{-1/2}\left(\frac{2p}{10\alpha - 1} + Cb^{m'}\right)^{-1} \tag{2.14}$$

$$t + t^* = \frac{\nu}{10\alpha}\int_b^{b_0} b^{-2}\left(\frac{2p}{10\alpha - 1} + Cb^{m'}\right)^{-2} db$$

$$m' = (1 - 10\alpha) / 20\alpha.$$

3. Let us now assume that the turbulence is isotropic. In this case the components of tensor $b_{i,j}(r_n, t)$ can be expressed in terms of a single scalar function $b_d^{d}(r, t)$ [1], of tensor $\langle u_i(x_n, t)\, u_j(x_n, t)\, u_k(x_n + r_n, t)\rangle$ in terms of a single scalar function $b_d^{nn}(r, t)$, and the components of tensor $\langle P(x_n, t)\, u_i(x_n + r_n, t)\rangle$ vanish. In these expressions $r = \sqrt{r_j r_j}$. Functions $b_d^{d}(r, t)$ and $b_d^{nn}(r, t)$ are related by the unique equation

$$\nu\left(\frac{\partial^2 b_d^{d}}{\partial r^2} + \frac{4}{r}\frac{\partial b_d^{d}}{\partial r}\right) - \frac{1}{2}\frac{\partial b_d^{d}}{\partial t} = \frac{\partial b_d^{nn}}{\partial t} + \frac{4}{r}b_d^{nn}$$

(the Kármán-Howarth equation) which corresponds to the tensor equation (1.1).

Let us assume that self-preservation of structural functions is observed with turbulence decay, i.e. there exist such functions l and b dependent on t and having the dimensions of length and of the square of velocity and such dimensionless functions β_2 and β_3 dependent on the dimensionless argument r / l, that for $r / l < \chi'$ the following relationships are satisfied:

$$b_d^{d}(r, t) = b_1(t) - b(t)\,\beta_2(r / l) \tag{3.2}$$

$$b_d^{nn}(r, t) = b^{3/2}(t)\,\beta_3(r / l)$$

Function $b_1(t)$ is determined by the equality $b_1(t) = b_d^d(0, t)$. In the particular case of $b(t) / b_1(t) = \text{const}$ hypotheses (3.2) convert to hypotheses of self-preservation of correlation functions.

Hypotheses (3.2) were first considered by Lin [6] without, however, finding all possible solutions.

Let us substitute expressions (3.2) into Eq. (3.1). Passing from variables r, t to $\chi = r / l$, t, we obtain

$$\frac{db_1}{dt} + \left[-\frac{db}{dt}\right]\beta_2 + \frac{2\nu b}{l^2}\left(\beta_2'' + \frac{4}{\chi}\beta_2'\right) + \frac{b}{l}\frac{dl}{dt}\chi\beta_2' = -\frac{2b^{3/2}}{l}\left(\beta_3' + \frac{4}{\chi}\beta_3\right) \qquad (3.3)$$

We divide all terms of Eq. (3.3) by $l^{-1} b^{3/2}$ and differentiate with respect to time t. This yields equation

$$(\omega_1(t)\cdot\omega_2(\chi)) = 0 \qquad (3.4)$$

$$\omega_1(t) = \left\{\left[lb^{-3/2}\frac{db_1}{dt}\right]_t', \ \left[-lb^{-3/2}\frac{db}{dt}\right]_t', \left[\frac{2\nu}{l\sqrt{b}}\right]_t', \left[b^{-1/2}\frac{dl}{dt}\right]_t'\right\}$$

$$\omega_2(\chi) = \{1, \beta_2, \beta_2'' + 4\beta_2' / \chi, \chi\beta_2'\}$$

which may be considered as showing that the scalar product of four-dimensional vectors is equal zero. These vectors whivh vary with variation of their arguments must, obviously, remain in fixed orthogonal planes (subspaces) of the four-dimensional space.

The following cases are possible.

1) $\omega_2(\chi)$ is an arbitrary vector. Then $\omega_1(t) = 0$, hence

$$b^{-1/2}\frac{dl}{dt} = c_1, \quad \frac{2\nu}{l\sqrt{b}} = c_2, \quad lb^{-3/2}\frac{db}{dt} = -c_3, \quad lb^{-3/2}\frac{db}{dt} = -c_4 \qquad (3.5)$$

Integration of this system of equations yields

$$l^2 = 4\nu\frac{c_1}{c_2}(t + t_0), \quad b = \frac{\nu}{c_1c_2}(t + t_0)^{-1}, \quad b_1 = \frac{c_4}{c_3}b + b_1^* \qquad (3.6)$$

With allowance for (3.5) from Eq. (3.3) we obtain for functions $\beta_2(\chi)$ and $\beta_3(\chi)$ a single equation with constant coefficients. To determine these functions we need additional assumptions.

This particular case was considered by Lin. Note that $b_1 \to b_1^*$ when $t \to \infty$. This means that for $b_1^* \neq 0$ the laws of turbulence decay (3.6) cannot be satistied for considerable t, while if $b_1^* = 0$, then $b(t) / b_1(t) = \text{const}$ and, consequently, hypotheses (3.2) convert to the Kármán-Howarth hypotheses about self-preservation of correlation functions.

2) $\omega_2(\chi)$ is a vector belonging to an arbitrary fixed three-dimensional plane, then $\omega_1(t)$ belongs to the one-dimensional straight line normal to that plane. This means that

$$-a_3 + a_2\beta_2 + a_1\chi\beta_2' + a_0(\beta_2'' + 4\beta_2' / \chi) = 0 \qquad (3.7)$$

and all functions of time in Eq. (3.4) are proportional to

$$-\frac{1}{a_3}\left[lb^{-3/2}\frac{db_1}{dt}\right]_t' = -\frac{1}{a_2}\left[lb^{-3/2}\frac{db}{dt}\right]_t' = \frac{1}{a_1}\left[b^{-1/2}\frac{dl}{dt}\right]_t' = \frac{1}{a_0}\left[\frac{2\nu}{l\sqrt{b}}\right]_t' \qquad (3.8)$$

It can be shown that $a_0 \neq 0$.

Let us assume that $a_0 = 0$. Since

$$\beta_2(\chi)|_{\chi=0} = \chi\beta_2'(\chi)|_{\chi=0} = 0$$

it follows from (3.7) that $a_3 = 0$. Comparing (3.4) and (3.8), we obtain

$$\left[lb^{-3/2}\frac{db_1}{dt}\right]_t' = \left[\frac{2\nu}{l\sqrt{b}}\right]_t' = 0$$

and, consequently,

$$-lb^{-3/2}\frac{db}{dt} = 2b^{-1/2}\frac{dl}{dt}$$

Then from (3.4) we obtain $2\beta_2 + \chi\beta_2' = 0$, hence $\beta_2 \sim \chi^{-2}$. But this solution does not satisfy condition $\beta_2(\chi)|_{\chi=0} = 0$. Hence $a_0 \neq 0$.

Let us set $a_0 = 2$. Then from (3.8) we obtain

$$b^{-1/2}\frac{dl}{dt} = a_1\frac{\nu}{l\sqrt{b}} + p, \quad lb^{-3/2}\frac{db}{dt} = -a_2\frac{\nu}{l\sqrt{b}} + q \tag{3.9}$$

$$lb^{-3/2}\frac{db_1}{dt} = -a_3\frac{\nu}{l\sqrt{b}} + s$$

Substituting equalities (3.9) into Eq. (3.3) and allowing for (3.7), we obtain

$$\beta_3' + \frac{4}{\chi}\beta_3 + \frac{s}{2} - \frac{q}{2}\beta_2 + \frac{p}{2}\chi\beta_2' = 0 \tag{3.10}$$

Since for small χ $\beta_3 \sim \chi^3$ (see [2]), hence Eq. (3.10) implies that $s = 0$. Thus in the second case Eqs. (3.7) and (3.10) for functions $\beta_2(\chi)$ and $\beta_3(\chi)$ and the system of equations (3.9) for $b_1(t)$, $b(t)$ and $l(t)$ with coefficients $a_0 = 2$ and $s = 0$ completely determine the law of turbulence decay.

3) Vectors $\omega_1(t)$ and $\omega_2(\chi)$ belong to two fixed and mutually perpendicular two-dimensional planes. This implies that function $\beta_2(\chi)$ must satisfy two linearly independent equations of the form (3.7). It can be shown that the solution of these equations has no physical meaning. Cases in which vector $\omega_2(\chi)$ belongs to a fixed straight line or is identically zero are, also, impossible.

Let us revert to the case (2).

Equation (3.7) for function $a_3 / a_2 - \beta_2(\chi)$ assumes the same form as the equation for the correlation function $f(\chi)$ derived in [2]. For $a_1 \neq 0$ the unique solution of that equation, which for $\chi = 0$ is a_3 / a_2, is a degenerate hypergeometric function [7], hence

$$\beta_2 = \frac{a_3}{a_2}\left\{1 - {}_1F_1\left[\frac{a_2}{2a_1}; \frac{5}{2}; -\frac{a_1\chi^2}{4}\right]\right\} \tag{3.11}$$

It can be readily shown (see, e.g. [3]) that

$$(-1)^{n+1}\frac{\partial^{2n}}{\partial r^{2n}}\beta_n(r,t)|_{r=0} = \frac{1}{b_1}\left\langle\left(\frac{\partial^n u_1}{\partial x_1^n}\right)^2\right\rangle > 0$$

From this and the expansion into Taylor series of ${}_1F_1$ (3.11) follows [7] that $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$.

Let us select the scales for $b(t)$ and $l(t)$ so as to have $a_1 = 0.5$ and $a_2 = a_3$. We introduce the notation $a_2 = 10\,\alpha$. Integration of Eq. (3.10) yields

$$\beta_3 = \frac{q}{10}\chi\beta_2 - \frac{5p+q}{10}\chi^{-4}\int_0^\chi \beta_2'\chi^5 d\chi \qquad (3.12)$$

Taking into account that $a_1 = 0.5$, $a_2 = a_3 = 10\,\alpha$, and $s = 0$, we find that the solution of the system of Eqs. (3.9) for $b_1(t)$ and functions $z(t)$ and $u(t)$ related to $b(t)$ and $l(t)$ by

$$u = l^{-1}b^{-1/2}, \quad z = l^{q/\gamma}b^{-p/\gamma} \qquad (3.13)$$

for $\alpha \neq 0.1$, $q \neq -2p$ and $q \neq -20\alpha p$ is of the form

$$u = C\frac{z}{\nu} + \frac{2p+q}{\nu(10\alpha-1)} \qquad (3.14)$$

$$t + t^* = \frac{2}{\nu(10\alpha-1)}\int z^{-1+\gamma'}u^{-4p/(2p+q)}dz$$

$$b_1 + b_1^* = \frac{20\alpha}{1-10\alpha}\int z^{-1-\gamma'}u^{-2q/(2p+q)}dz$$

$$\gamma = \frac{20\alpha p + q}{10\alpha - 1}, \quad \gamma' = \frac{2\gamma}{2p+q}$$

This solution has a meaning not for every arbitrary value of parameters α, p, q and C. First, since u and z are positive, C and $(2p+q)/(10\alpha-1)$ cannot be simultaneously negative and, second, for certain values of parameters solutions are determinate only for the limited time $t + t^* < 0$.

In all remaining cases the asymptotic laws of decay b, b_1, and l for $t \to \infty$ are:

1) for $C > 0$ and $\alpha > 0.1$

$$l \sim t^{1/2}, \quad b \sim b_1 + b_1^* \sim t^{-10\alpha}$$

2) for $p > |q/2|$ and $C(10\alpha - 1) < 0$

$$l \sim t^{2p/(2p-q)},\ b \sim t^{2q/(2p-q)},\ b_1 + b_1^* \sim t^{-1}$$

3) for $q < -20\alpha p$, $q < -2p$ and $\alpha < 0.1$

$$l \sim t^{1/2}, \quad b \sim b_1 + b_1^* \sim t^{-1}$$

Taking as basis the laws of decay $b(t)$, $b_1(t)$ and $l(t)$, we restate our conclusions as follows: if the hypotheses (3.2) about the self-conservation of structural functions β_2 and β_3 for $r/l < \chi'$ are satisfied in the case of decay of homogeneous and isotropic turbulent motions of a viscous incompressible fluid, then either b, b_1, and l vary in accordance with formulas (3.6) and functions $\beta_2(\chi)$ and $\beta_3(\chi)$ are related by a single equation so that supplementary assumptions are required for their determination, or the laws of decay $b(t)$, $b_1(t)$, and $l(t)$ are determined by the system of Eqs. (3.13) and (3.14), and the form of structural functions $\beta_2(\chi)$ and $\beta_3(\chi)$ is that defined by equalities (3.11) and (3.12).

4. On the assumption that $b(t) = b_1(t)$ the hypotheses (3.2) become the hypotheses of self-preservation of correlation functions f and h, i.e. of the existence of function $l(t)$ such that [1]

$$f(r, t) = f(\chi), \quad h(r, t) = h(\chi) \qquad (4.1)$$

$$(\chi = r/l, \quad b(t) = b_{ll}^{d}(0, t))$$

$$(b_{ll}^{d}(r, t) = bf(r, t), \quad b_{ll}^{nn}(r, t) = b^{3/2}h(r, t))$$

Sedov had analyzed and completely solved the problem of isotropic turbulence decay on the basis of these hypotheses [2].

The results obtained in Sect. 3 on the assumption that $b(t) = b_1(t)$ (hence in the first case $c_1 = c_3 = c_4$ and $b_1^* = 0$, and in the second $-a_2 = a_3$, $q = s = 0$ and $b_1^* = 0$), as well as the results obtained in Sect. 2 on the assumption of isotropic turbulence and the hypotheses (2.1) are satisfied throughout space k_n (hence in the second case $q = 0$ and in formula (2.13) $\psi_i = 0$ and $\psi = \text{const}$) are identical with the results of Sedov's investigations.

Taking as basis the laws of variation $b(t)$ and $l(t)$ we can formulate these results as follows: if the hypotheses (4.1) about the self-preservation of correlation functions f and h are satisfied in the case of homogeneous and isotropic turbulent motions of a viscous incompressible fluid, then either $b(t)$ and $l(t)$ (case 1) vary in accordance with formulas

$$l^2 = 4\nu \frac{c_1}{c_2}(t + t_0), \qquad b = \frac{\nu}{c_1 c_2}(t + t_0)^{-1} \tag{4.2}$$

and functions $f(\chi)$ and $h(\chi)$ are related by a single equation and their determination necessitates additional assumptions, or (case 2) the solution for $f(\chi)$ and $h(\chi)$ is given by formulas

$$f = {}_1F_1\left[10\alpha;\ \frac{5}{2};\ -\frac{\chi^2}{8}\right] \qquad (\alpha > 0) \tag{4.3}$$

$$h = \frac{p}{2}\chi^{-4}\int_0^\chi \chi^5 f'(\chi)\,d\chi = \frac{2\alpha p\chi}{4\alpha - 1}\left\{f - {}_1F_1\left[10\alpha + 1;\ \frac{7}{2};\ -\frac{\chi^2}{8}\right]\right\} \tag{4.4}$$

and the solution for $b(t)$ and $l(t)$ is determined by the system of equations (2.14).

Using the assumption that $b \to 0$ when $t \to \infty$ and substituting for the scale $l(t)$ the Taylor microscale $\lambda(t)$

$$\lambda^{-2} = -\left.\frac{\partial^2 f(r,t)}{\partial r^2}\right|_{r=0}, \qquad \lambda = l\sqrt{\alpha}, \qquad \frac{r}{\lambda} = \frac{\chi}{\sqrt{\alpha}} \tag{4.5}$$

we can express this solution in the following dimensionless form:

the first set $\alpha = \alpha_+$, $0 < w < \infty$

for $p(10\alpha - 1) > 0$, $C > 0$

$$\frac{\lambda^*}{\lambda} = \frac{2\sqrt{\alpha w}}{|10\alpha - 1|}(1 + w^{m'}) \tag{4.6}$$

$$\frac{\nu(t + t^*)}{\lambda^{*2}} = \frac{(10\alpha - 1)^2}{40\alpha}\int_w^\infty w^{-2}(1 + w^{m'})^{-2}\,dw$$

the second set $\alpha = \alpha_-$, $0 < w < 1$

for $C(10\alpha - 1) > 0$, $p < 0$

$$\frac{\lambda^*}{\lambda} = \frac{2\sqrt{\alpha w}}{1 - 10\alpha}(1 - w^{m'}) \tag{4.7}$$

$$\frac{\nu(t + t^*)}{\lambda^{*2}} = \frac{(10\alpha - 1)^2}{40\alpha}\int_w^{w_0} w^{-2}(1 - w^{m'})^{-2}\,dw$$

for $\alpha = 0.1$, $p < 0$

$$\frac{\lambda^*}{\lambda} = -\sqrt{\frac{w}{10}}\ln w \qquad (4.8)$$

$$\frac{\nu(t+t^*)}{\lambda^{*2}} = \int_w^{w_0} w^{-2}\ln^{-2} w\,dw$$

In formulas (4.6) – (4.8) the following notation is used:

$$w = \frac{b}{b^*}, \quad \lambda^* = \nu\sqrt{\frac{p^2}{b^*}}, \quad m' = \frac{1-10\alpha}{20\alpha} \qquad (4.9)$$

where t^* and b^* are constants of integration whose dimensions are, respectively, time and the square of velocity. Upper limits of integration w_0 of integrals in (4.7) and (4.8) have been chosen for definiteness so that $d\lambda\ /\ dw = d\lambda\ /\ dt = 0$ for $\nu\ (t + t^*)\ /\ \lambda^{*2} = 1$. For convenience the notation $\alpha = \alpha_+$ and $\alpha = \alpha_-$ is used for the first and second set, respectively.

The solution obtained for the second case, i.e. the complete solution for isotropic turbulence decay, derived by Sedov with the use of only the hypotheses of self-conservation of correlation functions, will be called the Sedov solution. This solution contains four independent parameters α, p, C and t^* (in formulas (4.3), (4.4) and (2.14)), or α (α_+ or α_-) b^*, λ^* and t^* (in formulas (4.3) – (4.9)), however, since the sets of functions

$$f\left(\frac{r}{\lambda}\right), \quad \frac{h}{p}\left(\frac{r}{\lambda}\right), \quad \frac{b}{b^*}\left[\frac{\nu(t+t^*)}{\lambda^{*2}}\right], \quad \frac{\lambda}{\lambda^*}\left[\frac{\nu(t+t^*)}{\lambda^{*2}}\right]$$

which determine the Sedov solution depend only on parameter α (α_+ or α_-), the latter is the sole significant parameter.

Functions of these sets are shown in Figs. 1 – 4 in the form of curves for $0.05 \leqslant \alpha \leqslant 0.2$. The dash lines in Figs. 3 and 4 relate to the asymptotic power laws for $t \to \infty$

$$\alpha > 0.1, \quad \lg\frac{b}{b^*} = 10\alpha\lg\frac{(10\alpha-1)^2}{4} - 10\alpha\lg\frac{\nu(t+t^*)}{\lambda^{*2}}$$

$$\lg\frac{\lambda}{\lambda^*} = \frac{1}{2}\lg\frac{\nu(t+t^*)}{\alpha\lambda^{*2}}$$

$$\alpha < 0.1, \quad \lg\frac{b}{b^*} = \lg\frac{(10\alpha-1)^2}{40\alpha} - \lg\frac{\nu(t+t^*)}{\lambda^{*2}}$$

$$\lg\frac{\lambda}{\lambda^*} = \frac{1}{2} + \frac{1}{2}\lg\frac{\nu(t+t^*)}{\lambda^{*2}}$$

The values of functions $f\ (r\ /\ \lambda)$ and $h\ (r\ /\ \lambda)$ corresponding to formulas (4.3) – (4.5) were calculated on a computer, using recurrent formulas and tabulated values of ${}_1F_1\ [a;\ 0.5;\ x]$ [7] for $\alpha = 0.05, 0.06, \ldots, 0.25$.

Sets

$$\frac{b}{b^*}\left[\frac{\nu(t+t^*)}{\lambda^{*2}}\right], \quad \frac{\lambda}{\lambda^*}\left[\frac{\nu(t+t^*)}{\lambda^{*2}}\right]$$

were calculated by formulas (4.5) – (4.9).

The integral in (4.8) reduces to an integral logarithm, and the integrals in (4.6) and (4.7) for rational α are expressed in terms of elementary functions.

For comparisons with experimental data the correlation function $g\ (r,\ t)$ and the one-dimensional spectrum $E_1(k_1, t)$, defined by

$$g = f + \frac{r}{2}\frac{\partial f}{\partial r}$$

$$b_d^{\,d}(r, t) = \int_0^\infty E_1(k_1, t) \cos k_1 r dk_1$$

may be necessary.

The definition of g and E_1 implies that the self-preservation $f(r, t) = f(\chi)$ is equivalent to self-preservation $g(r, t) = g(\chi)$ or $E_1(k_1, t) / bl = \varphi(k_1 l)$.

As regards the effect of variation of hypotheses (4.1) about the self-preservation of correlation functions f and h on derived results we note the following.

1) If hypotheses (4.1) are satisfied for $0 < r / l < \chi'$ (χ' is any positive number), the result is in complete agreement with that of Sedov.

2) If hypotheses (4.1) are satisfied for $\chi' < r / l < \chi''$ ($\chi' < \chi''$ are any positive numbers), then in the second case the variation reduces to an increase of the arbitrariness of integration of the system of equations for functions $f(\chi)$, $h(\chi)$, $b(t)$ and $l(t)$ derived by Sedov [2].

3) If weaker hypotheses about the constancy in time of parameters

$$\lambda^4(t)\frac{\partial^4}{\partial r^4} f(r, t)\big|_{r=0}, \quad \lambda^3(t)\frac{\partial^3}{\partial r^3} h(r, t)\big|_{r=0}$$

than (4.1) are satisfied, then b and $\lambda = l\sqrt{\alpha}$ vary with time in accordance with the system of equations (2.7) whose solutions are defined by formulas (4.6) – (4.8), if in the system (2.14) the constant of integration $C \neq 0$, or by formulas (4.2) if $C = 0$. This means that Sedov's conclusions about the law of decay of energy and of the linear scale of turbulence remain valid.

A reconsiliation of experimental data and Sedov's solution is given below. Owing to the insufficiency of experimental data no comparison is made of solutions derived in Sects. 2 and 3 with such data, it is, however, obvious that the use of these more general solutions can only improve the correlation between theory and experiment.

It is important to bear in mind that the laws of decay $b(t)$ and $l(t)$ for a homogeneous turbulence, when the hypotheses of self-preservation of spectral tensors (2.1) are satisfied in the whole space (or as can be shown, when the related hypotheses of self-preservation of correlation tensors $b_{i,j}$ and $S_{i,j}$ are satisfied at least in the neighborhood of some point $r_1 = r_2 = r_3 = 0$), are the same as the laws of decay $b(t)$ and $l(t)$ derived in Sedov's solution.

5. The majority of expreimental data on homogeneous and isotropic turbulence, including those cited in all references at the end of this paper, except [8, 9], relate to turbulent motions of air passed at constant velocity U through a grid of two mutually perpendicular rows of round bars of diameter d and spaced at pitch M.

Considerable deviations from isotropy [10, 11] and lack of tendency to isotropy buildup during turbulence decay were established. It was found that the ratio of mean square of longitudinal velocity pulsations to the mean square of transverse pulsations is greater than unity ($\langle u_1^2 \rangle / \langle u_2^2 \rangle = 1.3 \pm 0.2$).

In spite of these deviations from isotropy, Sedov's solution (in the second case) with parameters $\alpha_+ = 0.2$, $Ut^* / M = 0$, $U^2 / b^* = 700$ and $M / \lambda^* = 1.43$ is in good agreement with the experimental data of Batchelor and Townsend [12, 13] for

$R_M = UM / \nu = 650$ and $M / d = {}^{16}/_3$, as can be seen in Figs. 5 and 6. Small circles, crosses and black dots in Fig. 5 relate to experimental data on $f(r / \lambda)$ for $x / M = Ut / M = 320$, 640 and 960, respectively. In Fig. 6 black dots and crosses relate to experimental data on $b(t)$ for $M = 0.159$ cm and 0.635 cm, respectively, and the small circles to experimental data on $\lambda(t)$ for $M = 0.159$ cm. In both figures and remaining ones solid lines correspond to theoretical solutions.

When comparing experimental data with Sedov's solution, the parameters of the latter were determined as follows:

1) α was determined by comparing experimental points $f(r / \lambda)$ with the set of curves $f(r / \lambda; \alpha)$ calculated by (4.3) and (4.5);

2) λ^*, b^*, and the adherence to the $\alpha = \alpha_+$ or $\alpha = \alpha_-$ set was determined by comparing experimental points $\lambda(b)$ with a pair of theoretical curves (4.6) – (4.8)

$$\lg \frac{\lambda}{\lambda^*}\left(\lg \frac{b}{b^*}\right)$$

related to sets $\alpha = \alpha_+$ and $\alpha = \alpha_-$;

3) $t^* = x^* / U$ was determined by comparing experimental points $b(t)$ with the theoretical curve (4.6) – (4.8)

$$\frac{b}{b^*}\left[\frac{\nu(t+t^*)}{\lambda^{*2}}\right]$$

for known $\alpha = \alpha_+$ or $\alpha = \alpha_-$.

For the determination of all parameters of the solution it is sufficient to obtain measurements of b and λ at two different times and one value of $f(r / \lambda)$ for $r / \lambda \neq 0$. It is advisable to obtain the measured value of f in the interval $0.1 < f < 0.6$, because the curves of set $f(r / \lambda; \alpha)$ for small f, particularly when f is close to unity, lie very close to each other (formula (4.5) for λ implies that all curves $f(r / \lambda)$ have at point $f(0) = 1$ a common third order tangency with the parabola $f = 1 - r^2 / 2\lambda^2$).

We note that when $f(r, t)$ and $b(t)$ are known it is possible to determine $h(r, t)$ from the Kármán-Howarth equation (3.1). Hence the congruence of theoretical and experimental curves $b(t)$, $l(t)$ and $f(r / l)$ ensures that the experimental function h satisfies the self-preservation hypothesis and is in agreement with the curve theoretically predicted. In other words, a deviation of the theoretical curve $h(\chi)$ from the experimental $h(r, t)$ can be only due to the inaccurate fulfillment of the Kármán-Howarth equation and, consequently, of the assumptions as to the homogeneity and isotropy of the averaged motion and of the actual flow obeying the Navier-Stokes system of equations used in the derivation of that equation.

The agreement between theoretical curves $b(t)$, $\lambda(t)$ and $f(r / \lambda)$ and experimental points in Figs. 5 and 6 shows that, first of all, the hypotheses of self-preservation of correlation functions f and h (4.1) in the experiments of Batchelor and Townsend [12, 13] are satisfied for $R_M = 650$ and, second, that the second possible alternative considered in Sect. 4, i.e. the Sedov solution, is realized in their experiments.

Only one experiment carried out by Stewart [14] at $R_M = 5300$, $M / d = {}^{16}/_3$ and $M = 1{,}27$ cm in which the hypotheses (4.1) were directly verified was published. The scale $l_1 = M\sqrt{5 + Ut / M}$ was chosen by Stewart so that experimental points 1–5 of $f(r / l_1)$ (Fig. 7) and $h(r / l_1)$ (Fig. 8) obtained for $x / M = Ut / M = 20$, 30, 60, 90 and 120 in region $r / l_1 < 0.1$ fit a single curve. The marking of points is the same in Figs. 7 and 8.

Stewart, taking into consideration the experiments of Batchelor and Townsend, concluded that self-preservation of f is violated at small r (although this is not noticeable on the curve $f\ (r\ /\ l_1)$ in Fig. 7). The violation of self-preservation of f for small r makes it possible to use a wider class of solutions for the reconciliation of theory with experiment (see Note 2 in Sect. 4). In region $r\ /\ l_1 < 0.1$ (see Note 1 in Sect. 4) the Sedov solution with parameters

$$\alpha_- = 0.05,\ \ U^2\ /\ b^* = 400,\ \ M\ /\ \lambda^* = 20,\ \ \ Ut^*\ /\ M = -4,\ \ p = -0.071$$

is, also, in fair agreement with experimental data, as seen in Figs. 7 – 9.

Since Stewart had not published his data on $b\ (t)$, the experimental points of $b\ (t)$ shown in Fig. 9 are those obtained in [13] under similar conditions: $R_M = 5620$, $M\,/\,d = {}^{16}/_3$, $M = 0{,}635$ cm (crosses), and $M = 1.27$ cm (black dots). The small circles in Fig. 9 correspond to the scale l_1 used in Figs. 7 and 8. In reconciling Sedov's solution with experimental data $\lambda = l_1\ /\ 25$ was assumed.

With increasing Reynolds number R_M the correlation between Sedov's solution and experimental data on the shape of the correlation curve $f\ (r\ /\ \lambda)$ for turbulent motions of air downstream of a grid tends to deteriorate. Attempts by Batchelor and Townsend [13] to reconciliate this solution with their experimental data for $R_M = 11200$ and $M\ /\ d = 16/3$ proved to be negative (*).

The agreement between Sedov's solution with parameters $\alpha = 0.05,\ Ut^*\ /\ M = -9,\ U^2\ /\ b^* = 204$ and $M\ /\ \lambda^* = 39$ and data of a recent experiment [15] at $R_M = 17\ 000,\ M\ /\ d = {}^{16}/_3$ and $M = 2.54$ cm (Figs. 10 and 11) in which a motion very close to isotropic was obtained by slightly compressing the air downstream of the grid. A number of relationships between the component of the tensor of double correlations of velocity, which follow from the assumption of the motion isotropy, in particular the relationship $< u_1{}^2 > / < u_2{}^2 > = 1$ which was satisfied within 5%. In Fig. 10 black dots, small circles, straight and oblique crosses relate to the measured values of spectrum $E_1\ /\ b\lambda = \varphi\ (k_1\lambda)$ for $x\ /\ M = Ut\ /\ M = 45, 120, 240$ and 385 respectively. In Fig. 11 the black dots and the small circles relate to data on $b\ (t)$ and $\lambda\ (t)$, respectively.

Recently a number of experiments was carried out for determining the decay of turbulent motion of water downstream of various types of grids [8. 9], namely: (1) grid of the type described at the beginning of Sect. 5 with $M\ /\ d = 5$; (2) grid of the same type but with $M\ /\ d = 2.8$; (3) grid consisting of a single row of round bars to which a number of plates is rigidly fixed, oscillating around its axis. The maximum velocity of plate ends V_p in one experiment was three times and in the other seventeen times higher than the stream velocity U.

(*) These experimental data satisfy rather the hypothesis considered in Note 3 in Sect. 4 than the hypothesis about the self-preservation of correlation functions and the ensuing laws of variation of $b\ (t)$ and $\lambda\ (t)$ (4.7) (in system (2.14) $C \neq 0$). These laws, which are the same as Sedov's solution are in a considerably better agreement with experimental data than the laws (4.2) (in system (2.14) $C = 0$) accepted by Batchelor and Townsend and called by them the "initial" period of turbulence decay.

The main results of tests with these grids can be summarized as follows: (1) the parameter $\langle u_1^2\rangle / \langle u_2^2\rangle$ which defines anisotropy is equal unity within 5% ; (2) the hypothesis of self-preservation of f is very well satisfied and the curve $f(r/\lambda)$ is universal for all experiments. The empirical curve proposed by the authors virtually coincides with the curve (4.3) in the measurement region for $\alpha = 0.08$.

The values of parameters for Sedov's solution which correlate with experimental data [8, 9] as well as the notation used in Figs. 12 – 14 are shown in Table 1. For all experiments it was assumed $\alpha_- = 0.08$.

Table 1

Experimental conditions				Values of parameters			Marking of experimental points in Figs. 12 – 14
R_M	M/d	V_p/U	U,cm/sec	t^*, sec	U^2/b^*	M/λ^*	
470	2.8	—	2.0	−17	23.4	4.68	straight crosses
840	5	—	2.9	−11	72.4	11.5	oblique crosses
2000	—	3	3.14	−3	13.2	29.5	black dots
2000	—	17	3.14	3	0.4	50	small circles

The satisfactory correlation of experimental functions $b(t)$, $\lambda(t)$, and $f(r/\lambda)$ with theoretical ones can be observed in Figs. 12 – 14. As in experiments with air, this agreement must ensure the fulfillment of the self-preservation hypothesis for the experimental correlation function h and its conformity with the theoretical curve with the same degree of accuracy as the Kármán-Howarth equation is satisfied.

Thus, in experiments on perturbation decay in water downstream of various types of grids [8, 9], first, the hypotheses about the self-preservation of correlation functions f and h (4.1) are satisfied, second, the alternative (2) described in Sect. 4, i.e. Sedov's solution, is realized, and third, the only one significant theoretical parameter α is universal and $\alpha_- = 0.08$. Such universality does not exist in the definition of turbulent motions of air downstream of grids, even when these are geometrically similar. Thus in the experiments of Batchelor and Townsend [12, 13] α decreases from 0.2 to 0.05 when R_M is increased from 650 to 5620 and the solution passes from the set α_+ to the set α_-.

The qualitative divergence between experimental results with air and water is evident

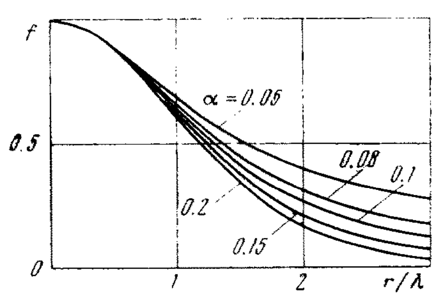

Fig. 1

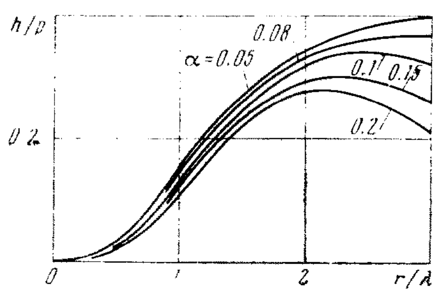

Fig. 2

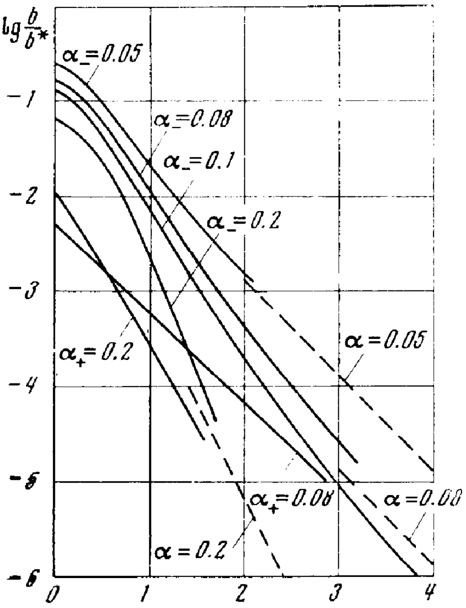

Fig. 3

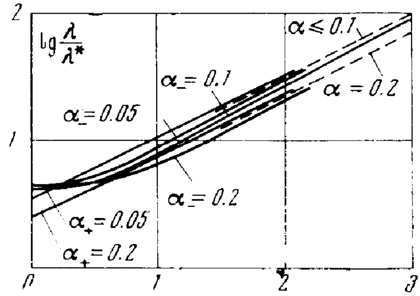

Fig. 4

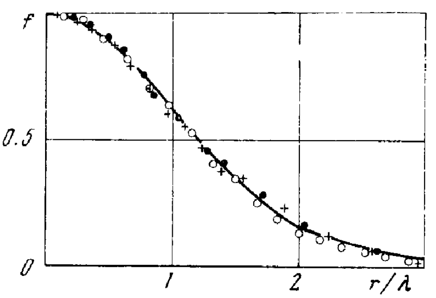

Fig. 5

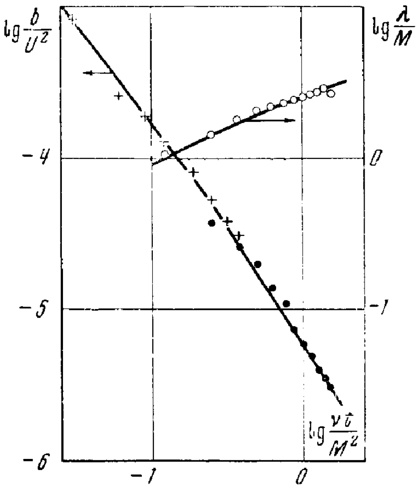

Fig. 6

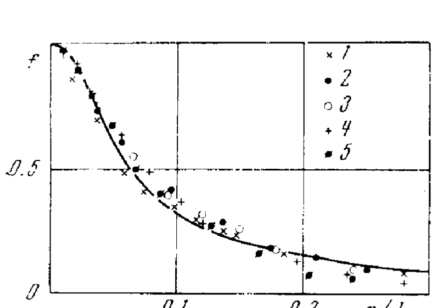

Fig. 7

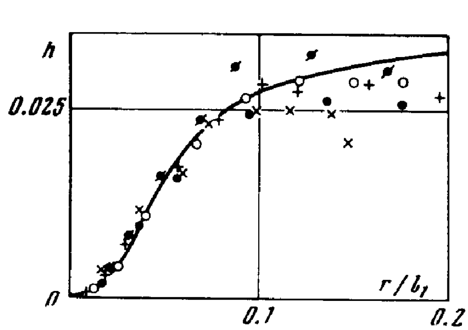

Fig. 8

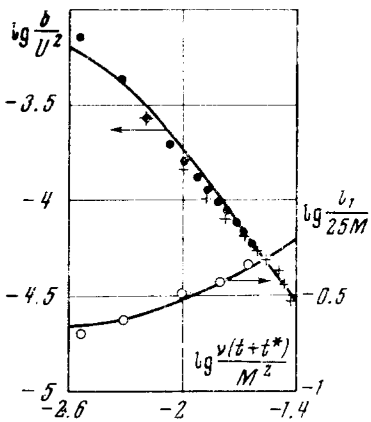

Fig. 9

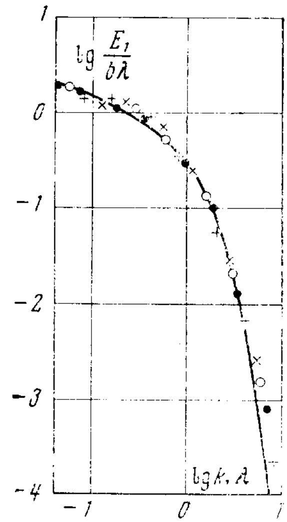

Fig. 10

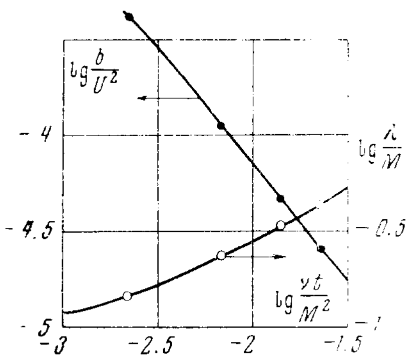

Fig. 11

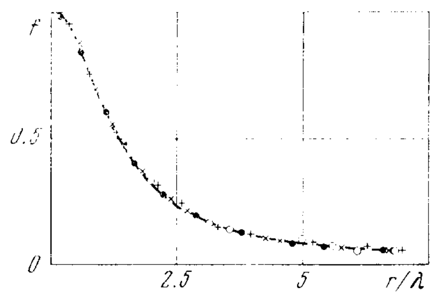

Fig. 12

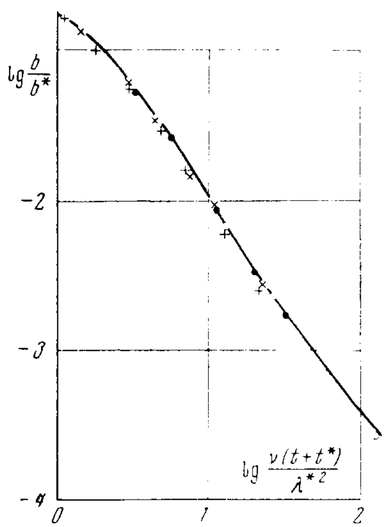

Fig. 13

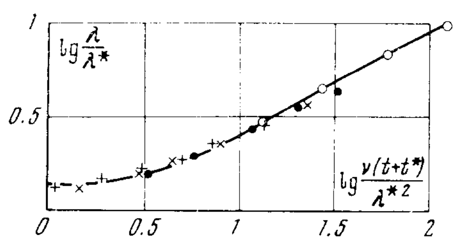

Fig. 14

from the comparison of correlation functions $f(r / \lambda)$ obtained under similar conditions by Batchelor and Townsend in experiments with air [12] ($R_M = 650$ and $M/d = {}^{16}/_{3}$) and Ling and Huang with water ($R_M = 840$ and $M/d = 5$) [8] and which were in agreement with the theoretical curve defined by (4.3) for $\alpha = 0.2$ and $\alpha = 0.08$, respectively, (see Figs. 1, 5 and 12).

Thus even for geometrically similar grids a divergence from similarity which depends on the Reynolds number $R_M = UM / \nu$ is observed in experiments (*). Hence it is not possible to establish universal relationships between parameters α, $\nu t^* / M^2$, U^2 / b^*, M / λ^* and the number R_M.

Nevertheless, as shown above, the Sedov solution derived from the Kármán-Howarth equation with the use of only hypotheses about the self-preservation of correlation functions and without resorting to hypotheses about the vanishing of the third or higher order correlations of velocity or on their dependence on secondary correlations (e.g. the hypotheses of Obukhov, Millionshchikov, Heisenberg, Kármán and Kovazhnyi), is in good agreement with experimental data on the decay of turbulent motion of air and water downstream of grids.

Finally, we note that in all cited experiments, except those of Batchelor and Townsend [12, 13] at $R_M = 650$, parameter $\alpha < 0.1$. For such α the asymptotic laws for $t \to \infty$ are : $b \sim t^{-1}$ and $\lambda \sim t^{1/2}$ and, consequently $\lambda \sqrt{b} / \nu \to \text{const} \neq 0$. This means that the so-called "final" period of turbulence decay, for which $\lambda \sqrt{b}/\nu \to 0$ when $t \to \infty$ and the terms of the Kármán-Howarth equations containing third moments

$$\frac{\partial b_d^{nn}}{\partial r} + \frac{4}{r} b_d^{nn}$$

become an infinitely small of a higher order than terms containing second moments [12], does not exist in the case of experiments defined by Sedov's solution with $\alpha < 0.1$.

The author thanks L. I. Sedov for suggesting this subject and his constant interest and valuable discussions on this work.

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*) Such divergence seems to take place at $x / M < 20$, so that in the region in which experimental data are compared with the model of isotropic turbulence ($x / M \geq 20$) the divergence of data in experiments with air and water may be attributed to initial conditions.

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CONTINUAL MECHANICS OF MONODISPERSE SUSPENSIONS. INTEGRAL AND DIFFERENTIAL LAWS OF CONSERVATION

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A disperse medium consisting of an incompressible fluid and small spheres of equal radii suspended in the fluid is considered as the superposition of two interpenetrating and interacting continua. Equations of conservation of mass, momentum and moment of momentum are obtained for the two continua in which all unknowns are expressed in terms of functionals of mean stresses acting at the surface of an individual suspended sphere.

The mathematical definition of the motion of a disperse system – investigated in numerous works – requires the solution of two distinct problems. The first of these consists of the formal derivation of "macroscopic" equations for the system phases which are assumed to be interpenetrating continuous media with specific properties. These equations which reflect the laws of conservation of mass, momentum and moment of momentum are usually obtained by known methods of mechanics of continuous media [1, 2]. The derivation of such equations for multiphase disperse systems of various kinds is treated, for instance, in [3 – 6]. However the obtained equations contain unknown